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A New Non-Trial-and-Error Method for Lag-Lead Compensator Design: A Special Case

Fei-Yue WANG

Abstract — **Based on the idea of Yeung-Wong-Chen's graphic-based non-trial-and-error method for three-parameter lag-lead compensator design and Wang's recent result on the exact and unique solution for single lag and lead compensator design, a new non-trial-and-error procedure is developed. The new approach is analytical and can be carried out by computer programs without manual graphical manipulations, thus provides the basis for more sophisticated design and performance analysis. Results obtained in this paper can be used to simplify and enhance current teaching and industrial practices for lag-lead compensator design in control and communication systems.**

Index Terms — **Lag-lead compensator, compensator design, control systems, trial-and-error method, analytic solution.**

I. Introduction

DUE to its simplicity and effectiveness, phase lag and lead compensation is essential for various frequency-based design methods, especially for design based on the Bode plots [1-20]. However, the nature of traditional trial-and-error graphical techniques currently used by almost all available textbooks [11-21] makes the learning and use of lag-lead compensation still a time consuming process. From 1970s to 1980s, various efforts have been made to develop analytical or computer-aided design procedures for lag and lead compensation [1-4] with limited success. Since 1995, Yeung *et al* [5-7] have made significant progress along this direction by constructing a universal design chart for non-trial-and-error design of lag-lead compensators. More recently, Calleja has proposed computer-based GUI tools to solve the problem of amplifier frequency compensation for input lag [8] and Wang [9-10] has obtained the exact and unique solution for single stage phase lag and lead compensation that can be used to construct pure analytical or non-trial-and-error computer programs to solve the lag-lead compensation problem.

In this paper, based on the idea of Yeung-Wong-Chen's graphic-based non-trial-and-error method for special lag-lead compensator design and Wang's recent result on single stage phase-lag and phase-lead compensation, a new and analytical non-trial-and-error approach is constructed for a special class of three-parameter lag-lead compensators. The results presented here is only part of our systematic effort to reformulate and reconstruct design procedures for frequency domain design methods. Later, we will deal with issues in general phase lag and lead compensation and corresponding sensitivity analysis for both specification parameters and solution procedures.

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Fei-Yue Wang is with the Key Laboratory of Complex Systems and Intelligent Science (LCSIS), Institute of Automation, the Chinese Academy of Sciences, Beijing 100080, China, and the Program for Advanced Research in Complex Systems (PARCS), the University of Arizona, Tucson, AZ 85721, USA (E-mail: feiyue@sie.arizona.edu)

II. Lag-Lead Compensators

Consider a lag-lead compensator in the form [4, 11-20]

$$G_c(s) = K_c \frac{1+\alpha\tau s}{1+\tau s}\cdot\frac{1+\beta\sigma s}{1+\sigma s} = K_c \bar{G}_c(s) \qquad (1)$$

where $\tau > 0,\ \sigma > 0,\ \alpha > 0,\ \beta > 0.$ and $\alpha\beta = 1$.

From Eq. (1), it is obvious that if $(\tau, \sigma, \alpha, \beta)$ is a valid solution for the lag-lead compensation, then $(\sigma, \tau, \beta, \alpha)$, $(\tau, \sigma, \beta\sigma/\tau, \alpha\tau/\sigma)$ and $(\sigma, \tau, \alpha\tau/\sigma, \beta\sigma/\tau)$ are also valid solutions, therefore, the solution for any design specification is not unique; this is discussed further in Sections III and IV.

Substitute s with $j\omega$ in Eq. (1), we have,

$$\bar{G}_c(j\omega) = \frac{1+j\Delta(\omega)\Gamma}{1+j\Delta(\omega)} \qquad (2)$$

where,

$$\Gamma = \frac{\alpha\tau+\beta\sigma}{\tau+\sigma},\ \Delta(\omega) = \frac{(\tau+\sigma)\omega}{1-\tau\sigma\omega^2} \qquad (3)$$

As usual, K_c is determined from the steady-state accuracy specification using the final value theorem for *Laplace* transforms [4, 11-20]. To determine other three parameters, consider the specified gain of compensator at a given frequency ω:

$$\left|\bar{G}_c(j\omega)\right| = c,\ \angle\bar{G}_c(j\omega) = p \qquad (4)$$

where c and p are the required gain (in dB) and phase (in rad,) respectively. From Eq. (2), the specified gain leads to the following equations,

$$\frac{1+(\Delta\Gamma)^2}{1+\Delta^2} = c^2,\ \tan^{-1}(\Delta\Gamma) - \tan^{-1}(\Delta) = p \qquad (5)$$

Using similar transformations and solution procedures developed in our previous work [9], we find that (the detailed derivation is given in the Appendix at the end of paper),

$$\Gamma(c,\delta) = \frac{c\left(c\sqrt{1+\delta^2}-1\right)}{c-\sqrt{1+\delta^2}},\ \Delta(c,\delta) = \frac{c-\sqrt{1+\delta^2}}{c\delta} \qquad (6)$$

where $\delta = \tan(p)$.

Clearly, a lag-lead design is feasible only if one of the following two conditions holds:

1) $c < 1,\ c\sqrt{1+\delta^2} < 1$; In this case, we find,

$$0 < \Gamma < \left(\sqrt{1+\delta^2} - |\delta|\right)^2 < 1/\left(1+\delta^2\right) < 1;$$

$$\Delta < 0 \text{ if } \delta > 0,\ \Delta > 0 \text{ if } \delta < 0.$$

2) $c > 1$, $c > \sqrt{1+\delta^2}$; In this case, we find,

$$p > \left(\sqrt{1+\delta^2} + |\delta|\right)^2 > 1+\delta^2 > 1;$$

$$\Delta > 0 \text{ if } \delta > 0,\ \Delta < 0 \text{ if } \delta < 0.$$

Based on Eqs (1-6) and conditions 1) and 2), we are already to discuss the new analytical non-trial-and-error design method for lag-lead compensators.

III. Basic Performance Specifications

Fig.1 shows a compensation system, where plant and compensator are represented by $G_p(s)$ and $G_c(s)$.

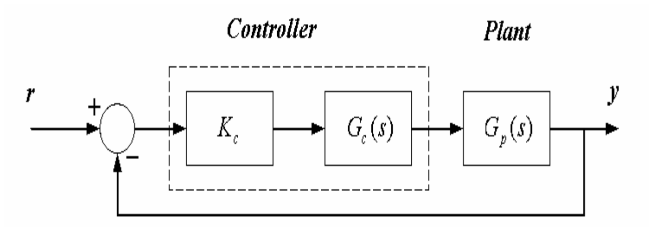

Fig. 1: A Typical Feedforward Compensation System

For a desired gain margin GM at a phase crossover frequency ω, we have,

$$\left|G_c(j\omega)G_p(j\omega)\right|_{dB} = -GM,\ \angle\left\{G_c(j\omega)G_p(j\omega)\right\} = -180°$$

where $\left|(\cdot)\right|_{dB} = 20\log_{10}(\cdot)$, or in terms of compensator only,

$$\begin{aligned}\left|\overline{G}_c(j\omega)\right|_{dB} &= 20\log_{10} c = -\left|K_c G_p(j\omega)\right| - GM,\\ \angle\overline{G}_c(j\omega) &= p = -\angle G_p(j\omega) - 180^o\end{aligned} \tag{7}$$

Similarly, for a desired phase margin PM at a gain crossover frequency ω, we have,

$$\left|G_c(j\omega)G_p(j\omega)\right|_{dB} = 0,\ \angle\left\{G_c(j\omega)G_p(j\omega)\right\} = PM - 180°$$

or in terms of the compensator only,

$$\begin{aligned}\left|\overline{G}_c(j\omega)\right|_{dB} &= 20\log_{10} c = -\left|K_c G_p(j\omega)\right|,\\ \angle\overline{G}_c(j\omega) &= p = PM - \angle G_p(j\omega) - 180^o\end{aligned} \tag{8}$$

Now, let's first consider the following dual basic performance specifications for compensator design,

B1) *Design a lag-lead compensator for the specified gain margin GM at phase crossover frequency* ω_1, *and the specified gain crossover frequency* ω_2;

B2) *Design a lag-lead compensator for the specified phase crossover frequency* ω_1, *and the specified phase margin PM at gain crossover frequency* ω_2.

Note that in **B1**), no phase margin is specified, while in **B2**) no gain margin is specified.

The design procedure for specification **B1)** can be established as follows,

Procedure B1:

B1-1) Calculate c_1, p_1, and c_2;

$$c_1 = \frac{10^{-GM/20}}{K_c\left|G_p(j\omega_1)\right|},\ c_2 = \frac{1}{K_c\left|G_p(j\omega_2)\right|},$$
$$p_1 = -\angle G_p(j\omega_1) - 180°; \tag{9}$$

and

$$p_2 = PM - \angle G_p(j\omega_2) - 180°,$$

is unknown since PM is not specified here. Define,

$$\delta_i = \tan(p_i),\ \Gamma_i = \Gamma(c_i, \delta_i),\ \Delta_i = \Delta(c_i, \delta_i);$$

B1-2) Find δ_2; From Eq. (3), since Γ is independent of frequency, we must have $\Gamma = \Gamma_1 = \Gamma_2$, which leads to,

$$\delta_2 = \pm\sqrt{c_2(1+\Gamma)/\left(c_2^2+\Gamma\right) - 1} \tag{10}$$

Clearly, a solution is possible only when

$$\min(1,\Gamma) < c_2 < \max(1,\Gamma)$$

B1-3) Calculate Γ, Δ_1, Δ_2 using Eq. (6);

B1-4) Find $(\tau, \sigma, \alpha, \beta)$ using Eq. (3);

Since

$$\tau\sigma = Q_1,\ \tau + \sigma = Q_2$$

where

$$Q_1 = \frac{\omega_{12} - \Delta_{12}}{(1-\omega_{12}\Delta_{12})\omega_1\omega_2},\ Q_2 = \frac{\left(1-\omega_{12}^2\right)\Delta_1\Delta_2}{\omega_1\Delta_2 - \omega_2\Delta_1\omega_{12}^2}, \tag{11}$$

$$\omega_{12} = \omega_1/\omega_2,\ \Delta_{12} = \Delta_1/\Delta_2$$

and

$$\alpha\beta = 1,\ \alpha\tau + \beta\sigma = Q_3$$

where

$$Q_3 = (\tau + \sigma)\Gamma = Q_2\Gamma \tag{12}$$

It follows that,

$$\tau = \left(Q_2 \pm \sqrt{Q_2^2 - 4Q_1}\right)/2,\ \sigma = \left(Q_2 \mp \sqrt{Q_2^2 - 4Q_1}\right)/2 \tag{13}$$

$$\alpha = \left(Q_3 \pm \sqrt{Q_3^2 - 4Q_1}\right)/2\tau,\ \beta = \left(Q_3 \mp \sqrt{Q_3^2 - 4Q_1}\right)/2\sigma \tag{14}$$

Note that there are four possible combinations for each value of δ_2, but they actually represent the same compensator, as we have discussed in the previous section.

B1-5) Verify the condition for positive coefficients:

$$\tau > 0, \sigma > 0, \alpha > 0, \beta > 0$$

The design procedure for specification **B2**) is almost identical to that of **B1**), that is,

Procedure B2:

B2-1) Calculate p_1, p_2, and c_2;

$$p_1 = -\angle G_p(j\omega_1) - 180°,$$
$$p_2 = PM - \angle G_p(j\omega_2) - 180°, \tag{15}$$
$$c_2 = \frac{1}{K_c\left|G_p(j\omega_2)\right|},\ c_1 = \frac{10^{-GM/20}}{K_c\left|G_p(j\omega_1)\right|}$$

and c_1 is unknown since GM is not specified here.

B2-2) Find c_1; From Eq. (3), since $\Gamma = \Gamma_1 = \Gamma_2$, thus,

$$c_1 = \frac{1+\Gamma \pm \sqrt{(1-\Gamma)^2 - 4\delta_1^2\Gamma}}{2\left(1+\delta_1^2\right)} \tag{16}$$

Clearly, a solution is possible only when

$$\delta_1^2 < (1-\Gamma)^2/(4\Gamma)$$

B2-3) to **B2-5**) is same as **B1-3**) to **B1-5**).

To illustrate both design procedures, we consider the same example for the lag-lead compensation by Yeung *et al* in [4], where plant is given as $G_p(s) = 100/\left[s(s+5)(s+10)\right]$, and

Fig. 2 shows its Bode plots without compensation. For the steady state error, the specification is given in terms of the required static velocity error constant $K_v = 100/\sec$. Based on this condition, it is easy to find K_c as,

$$K_v = \lim_{s\to 0} sG_c(s)G_p(s) = 2K_c = 100,$$

Therefore, $K_c = 50$.

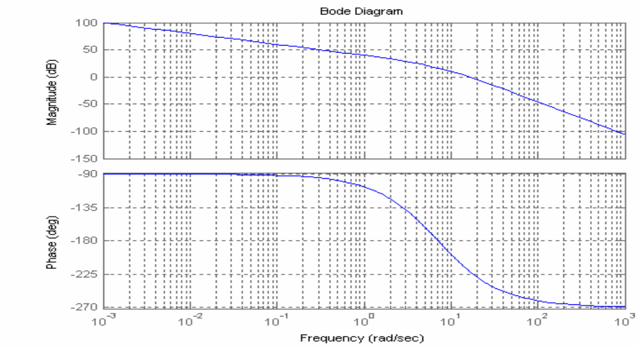

Fig. 2: Bode Plots for Uncompensated Plant

***Design Example for Specification* B1)**

In this case, the design problem is given as: find a lag-lead compensator for the specified gain margin $GM = 12\ dB$ at phase crossover frequency $\omega_1 = 18.3\ rad$, and the specified gain crossover frequency $\omega_2 = 8.5\ rad$.

Step 1): It is easy to find that from Eq. (9),

$$c_1 = 0.3637,\ \delta_1 = 1.0379,\ c_2 = 0.2200;$$

Step 2): δ_2 is found by Eq. (10) as,

$$\delta_2 = \pm 0.7019$$

Step 3): $\Delta_1, \Delta_2, \Gamma$ is calculated from Eq. (6),

$$\Delta_1 = -2.8545,\ \Delta_2 = \pm 6.4865,\ \Gamma = 0.1606$$

Step 4): Find all solutions using Eqs. (13-14),

For $\delta_2 = 0.7019$, the four solutions are,

$$[\tau, \sigma, \alpha, \beta] = \begin{bmatrix} 10.7694 & 0.0195 & 0.1488 & 6.7225 \\ 10.7694 & 0.0195 & 0.0121 & 82.3427 \\ 0.0195 & 10.7694 & 82.3427 & 0.0121 \\ 0.0195 & 10.7694 & 6.7225 & 0.1488 \end{bmatrix}$$

For $\delta_2 = -0.7019$, the four solutions are,

$$[\tau, \sigma, \alpha, \beta] = \begin{bmatrix} 0.2580 & 0.0332 & 0.0906+0.3470j & 0.7047-2.6979j \\ 0.2580 & 0.0332 & 0.0906-0.3470j & 0.7047+2.6979j \\ 0.0332 & 0.2580 & 0.7047+2.6979j & 0.0906-0.3470j \\ 0.0332 & 0.2580 & 0.7047-2.6979j & 0.0906+0.3470j \end{bmatrix}$$

Step 5): Verify the solution; clearly, only the first four solutions corresponding to $\delta_2 = 0.7019$ are valid. Actually, as discussed previously in Section 2, all four solutions represent a single compensator. Therefore, the final unique solution for this design example is,

$$\bar{G}_c(s) = \frac{(s+0.6242)(s+7.6460)}{(s+0.0929)(s+51.4004)} = \frac{s^2+8.2702s+4.7727}{s^2+51.4932s+4.7727}$$

The corresponding phase margin is $25.1645°$. The Bode plots for the compensated system is given in Fig. 3.

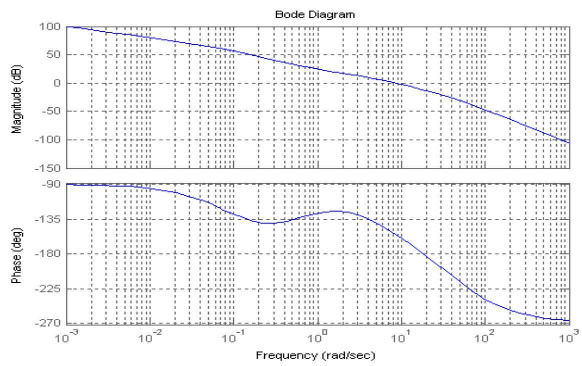

Fig. 3: Bode Plots for Compensated Plant in Specification **B1**)

***Design Example for Specification* B2)**

In this case, the design problem is given as: find a lag-lead compensator for the specified gain margin $PM = 25°$ at gain crossover frequency $\omega_2 = 8.5\ rad$, and the specified phase crossover frequency $\omega_1 = 18.3\ rad$.

Step 1): It is easy to find that from Eq. (15),

$$\delta_1 = 1.0379,\ c_2 = 0.2200,\ \delta_2 = 0.6976$$

Step 2): c_1 is found by Eq. (16) as,

$$c_1 = 0.4367 \text{ or } c_1 = 0.3690$$

Step 3): $\Delta_1, \Delta_2, \Gamma$ is calculated from Eq. (6),

$$\Delta_1 = -2.2165 \text{ or } \Delta_1 = -2.8001,\ \Delta_2 = -6.5103,\ \Gamma = 0.1611$$

Step 4): Find all solutions using Eqs. (13-14),

For $c_1 = 0.4367$, the four solutions are invalid,

$$\begin{bmatrix} 1.6221 & 0.0269 & 0.0819 + 0.0993j & 4.9378 - 5.9926j \\ 1.6221 & 0.0269 & 0.0819 - 0.0993j & 4.9378 + 5.9926j \\ 0.02690 & 1.6221 & 4.9378 + 5.9926j & 0.0819 - 0.0993j \\ 0.02690 & 1.6221 & 4.9378 - 5.9926j & 0.0819 + 0.0993j \end{bmatrix}$$

For $c_1 = 0.3690$, the four solutions are valid,

$$\begin{bmatrix} 7.4959 & 0.0200 & 0.1429 & 6.9973 \\ 7.4959 & 0.0200 & 0.0186 & 53.6557 \\ 0.0200 & 7.4959 & 6.9973 & 0.1429 \\ 0.0200 & 7.4959 & 53.6557 & 0.0186 \end{bmatrix}$$

Step 5): Verify the solutions; clearly, only the second four solutions corresponding to $c_1 = 0.3690$ are valid and represent a single compensator,

$$\bar{G}_c(s) = \frac{(s+0.9335)(s+7.1580)}{(s+0.1334)(s+50.2200)} = \frac{s^2+8.0915s+6.6819}{s^2+50.2200s+6.6819}$$

The gain margin from this compensator is $11.8753\ dB$, very close to $12\ dB$ in the previous design example. Therefore, the result for this design problem is also close to that of the previous design example, as expected.

IV. CLASSIC PERFORMANCE SPECIFICATIONS

The basic design performance specifications discussed in the last section are different from the classic performance specifications used in most of control textbooks [11-20], which can stated as followings,

C1) *Design a lag-lead compensator for the specified gain margin GM at phase crossover frequency* ω_1, *and the specified phase margin* PM;

C2) *Design a lag-lead compensator for the specified gain margin GM, and the specified phase margin PM at gain crossover frequency ω_2.*

In **C1**), ω_2 is not specified, so neither c_2 nor δ_2 can be calculated, and in **C2**) ω_1 is not specified, so neither c_1 nor δ_1 can be calculated. In both cases, no pure analytic solution as in **B1**) or **B2**) is available. However, a simple search procedure can be implemented analytically to find the solution within a specified range of gain crossover frequency or phase crossover frequency.

For a given range $[\omega_{2\min}, \omega_{2\max}]$ of gain crossover frequency, the design procedure for specification **C1**) can be stated as following.

Procedure C1:

C1-1) Calculate c_1, and δ_1;

$$c_1 = \frac{10^{-GM/20}}{K_c\left|G_p(j\omega_1)\right|},\quad p_1 = -\angle G_p(j\omega_1) - 180°;$$

C1-2) Search for ω_2; for any ω_2 in $[\omega_{2\min}, \omega_{2\max}]$, calculated c_2 using Eq. (9) and δ_2 using Eq. (10), and then find the corresponding phase margin as,

$$PM(\omega_2) = 180° + \angle G_p(j\omega_2) + p_2(\omega_2)$$

Search ω_2 that minimizes the error in the phase margin,

$$\omega_2 = \arg\left\{\min_{\omega_{2\min}<\omega_2<\omega_{2\max}} \left|PM - PM(\omega_2)\right|\right\}$$

The search process can be carried out using the standard *Matlab* function *Fmin*.

C1-3) to **C1-5**) is the same as **B1-3**) to **B1-5**).

Similarly, for a given range $[\omega_{1\min}, \omega_{1\max}]$, the design procedure for specification **C2**) can be stated as,

Procedure C2:

C2-1) Calculate c_2, and δ_2;

$$c_2 = \frac{1}{K_c\left|G_p(j\omega_2)\right|},\quad p_2 = PM - \angle G_p(j\omega_2) - 180°$$

C2-2) Search for ω_1; for any ω_1 in $[\omega_{1\min}, \omega_{1\max}]$, calculated δ_1 using Eq. (15) and c_1 using Eq. (16), and then find the corresponding gain margin as,

$$GM(\omega_1) = -\frac{10^{-c_1/20}}{K_c\left|G_p(j\omega_1)\right|}$$

Search ω_1 that minimizes the error in the gain margin

$$\omega_1 = \arg\left\{\min_{\omega_{1\min}<\omega_1<\omega_{1\max}} \left|GM - GM(\omega_1)\right|\right\}$$

The search process can be carried out using *Fmin*.

C2-3) to **C2-5**) is same as **C1-3**) to **C1-5**).

To illustrate both design procedure, we consider the same design example as in the previous section.

Design Example for Specification C1)

In this case, the design problem is given as: find a lag-lead compensator for the specified gain margin $GM = 11.6127\ dB$ at phase crossover frequency $\omega_1 = 20.65\ rad$, and the specified phase margin $PM = 41.7646°$.

Step 1): It is easy to find that from Eq. (9),

$$c_1 = 0.5288,\ \delta_1 = 1.2152;$$

Step 2): Search in the range of $[5,10]\ rad/s$, ω_2 is found as, $\omega_1 = 9.500$, and corresponding $c_2 = 0.2813$, $\delta_2 = 1.5688$.

Step 3): $\Delta_1, \Delta_2, \Gamma$ is calculated from Eq. (6),

$$\Delta_1 = -1.6262,\ \Delta_2 = -3.5776,\ \Gamma = 0.0849;$$

Step 4) and *Step* 5): the desired compensator is found to be,

$$(\tau, \sigma, \alpha, \beta) = (24.5389\ 0.0299\ 0.0668\ 14.9793);$$

and the corresponding transfer function is,

$$\bar{G}_c(s) = \frac{(s+0.6104)(s+2.2320)}{(s+0.0408)(s+33.4338)} = \frac{s^2+2.8424s+1.3625}{s^2+33.4745s+1.3625}$$

Fig. 4 presents the Bode plot for the compensated system, and Fig. 5 shows phase margin as a function of frequency for the given range of gain crossover frequency ω_2. Clearly, there are two frequencies that meet the phase margin requirement, one is 5.80, and the other is 9.50, which has been found by our procedure. Obviously, we can modify step **C1-2**) to find all admissible gain crossover frequencies and the corresponding compensators, if they are valid.

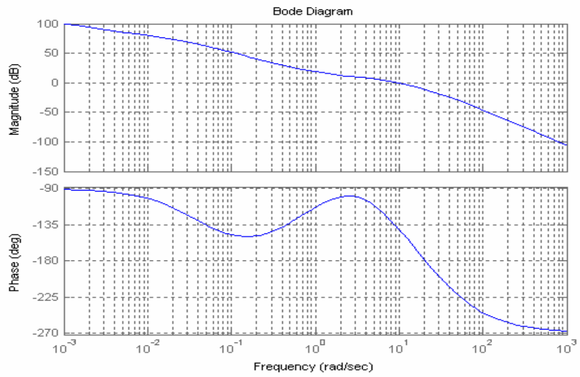

Fig. 4: Bode Plots for Compensated Plant in Specification **C1**)

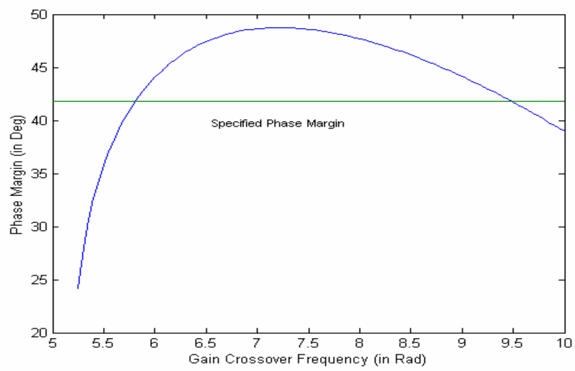

Fig. 5: Phase Margin as a Function of Frequency for Specification **C1**)

Design Example for Specification C2)

In this case, the design problem is given as: find a lag-lead compensator for the specified gain margin $GM = 12\ dB$ and phase margin $PM = 42°$ at a gain crossover frequency $\omega_2 = 9\ rad$. Note that this example is identical to the design problem discussed by Yeung *et al* in [4].

Step 1): It is easy to find that from Eq. (15),

$$c_2 = 0.2493,\ \delta_2 = 1.4246;$$

Step 2): Search in the range of $[15, 24]$ rad/s, ω_1 is found as, $\omega_1 = 20.6700$, and the corresponding $c_1 = 0.5092$, $\delta_1 = 1.2167$.

Step 3): $\Delta_1, \Delta_2, \Gamma$ is calculated from Eq. (6),

$$\Delta_1 = -1.7204,\ \Delta_2 = -4.1984,\ \Gamma = 0.0946 \qquad (17)$$

Step 4) and *Step* 5): the desired compensator is found to be,

$$(\tau, \sigma, \alpha, \beta) = \left(0.0286\ \ 6.0116\ \ 9.9783 - 10.5024j\ \ 0.0475 + 0.0500j\right); \qquad (18)$$

Clearly, this is not a valid lag-lead compensator, however the corresponding combined transfer function is,

$$\bar{G}_c(s) = \frac{s^2 + 3.3197s + 5.8072}{s^2 + 35.0767s + 5.8072} \qquad (19)$$

which is a valid controller for feedforward compensation (but not a lag-lead compensator!). According to [4], the desired compensator in this case is,

$$\bar{G}_c(s) = \frac{(s + 0.987)(s + 2.5)}{(s + 0.0675)(s + 36.54)} = \frac{s^2 + 3.487s + 2.4675}{s^2 + 36.6075s + 2.4665} \qquad (20)$$

Note that the specified information and obtained information for both design examples are very close, but the final solutions are quite different. The step responses of the closed system in the time domain for compensator (19), the compensator found in **C1**), and compensator (20) are given in Fig. 4. Clearly, both perform similarly and well, and their Bode plots are almost identical. In the following section, we discuss in detail why this is happening.

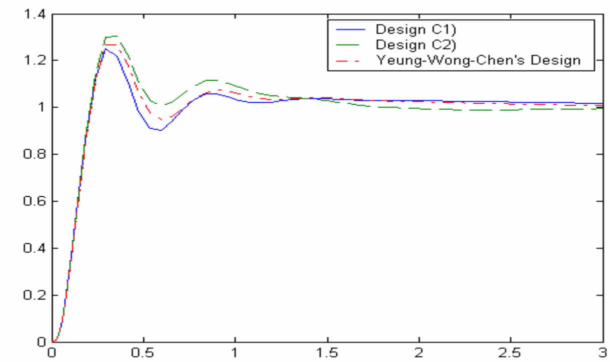

Fig. 4: Step Responses of Compensated Systems for Specs. **C1**) and **C2**)

Fig. 7 shows gain margin for the given range of phase crossover frequency ω_1. Note that in this case only one frequency meets the requirement for gain margin.

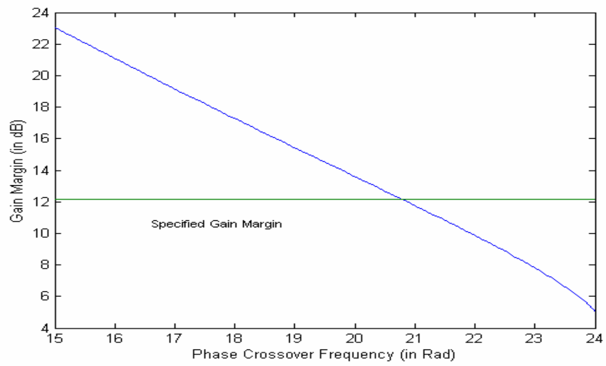

Fig. 7: Gain Margin as a Function of Frequency for Specification **C2**)

V. Relation to Yeung-Wong-Chen Procedure

The numerical results of design example for specification **C2**) in the previous section indicate that the solutions for lead-lag compensators from the current approach and the non-trial-and-error graphical method proposed by Yeung *et al* in [4] can be quite different for the same specification. To find out the reason for the difference, we investigate the relationship between the two approaches here.

Yeung-Wong-Chen Procedure can be summarized as:

$$T_{1,2} = B \pm \sqrt{B^2 - C} \qquad (21)$$

$$\gamma = \frac{\theta(T_1 + T_2) + \sqrt{\theta^2 (T_1 + T_2)^2 - 4T_1T_2}}{2T_2}$$

where,

$$B = \frac{\Omega_1\Omega_2(\omega_1^2 - \omega_2^2)}{\omega_1\omega_2(\omega_1\Omega_1 - \omega_2\Omega_2)},\ C = \frac{\omega_2\Omega_1 - \omega_1\Omega_2}{\omega_1\omega_2(\omega_1\Omega_1 - \omega_2\Omega_2)}$$

$$\Omega_i = \frac{(T_1 + T_2)\omega_i}{2(1 - T_1T_2\omega_1^2)},\ \theta = \frac{\gamma T_2 + T_1}{\gamma(T_1 + T_2)} \qquad (22)$$

and Ω, θ, and ω are obtained manually from the universal design chart for lag-lead compensators proposed in [4]. Note ω_1 in [4] is ω_2 here, and ω_2 in [4] is ω_1 here.

Comparing Yeung-Wong-Chen Procedure and the current approach, we found Eq. (2) and,

$$\Gamma = 1/\theta,\ \Delta = 2\theta\Omega$$

After tedious derivation and algebraic manipulations, one can show that Eq. (21) indeed leads to Eqs. (13-14), and *vice versa*. Therefore, mathematically, there is no difference between the two approaches. The only difference between the two is numerical and depends on the way employed to determine $(\Omega_1, \Omega_2, \theta, \omega_1)$ and $(\Delta_1, \Delta_2, \Gamma, \omega_1)$: a graphical and manual approach has been used to find $(\Omega_1, \Omega_2, \theta, \omega_1)$ in this paper. This can be seen clearly from the numerical example provided in [4].

For the design example for specification **C2**), the solution found in [4] is,

$$\Omega_1 = -0.085,\ \Omega_2 = -0.2,\ \theta = 10.5,\ \omega_1 = 20.65;$$

$$T_1 = 0.4,\ T_2 = 1.013,\ \gamma = 14.167$$

or equivalently,

$$\Delta_1 = -1.7850,\ \Delta_2 = -4.20000,\ \Gamma = 0.0953,\ \omega_1 = 20.65$$

$$[\tau, \sigma, \alpha, \beta] = [0.0282\ \ 14.3512\ \ 14.1670\ \ 0.0706]$$

which is different from (17) and (18) in the previous section. However, when applying $(\Delta_1, \Delta_2, \Gamma, \omega_1)$ into Eqs. (13-14), we find that,

$$[\tau, \sigma, \alpha, \beta] = [0.0273\ \ 15.9489\ \ 13.9865\ \ 0.0715]$$

which are close enough to the equivalent solution.

Now, transform

$$(\Delta_1, \Delta_2, \Gamma, \omega_1) = (-1.7204,\ -4.1984,\ 0.0946,\ 20.67)$$

in the previous section for the design example of **C2**) into the equivalent values in Yeung-Wong-Chen's method, we find,

$$(\Omega_1, \Omega_2, \theta, \omega_1) = (-0.0814,\ -0.1986,\ 10.5708,\ 20.67)$$

$$[T_1\ \ T_2\ \ \gamma] = [0.2854 + 0.3004j\ \ 0.2858 - 0.3008j\ \ 9.9783 + 10.5024j],$$

and when applying $(\Omega_1, \Omega_2, \theta, \omega_1)$ into Eq. (21), we have,

$$[T_1\ T_2\ \gamma] =$$
$$[0.2810+0.3010j\ \ 0.2810-0.3010j\ \ 9.7990+10.4951j]$$

Again, close enough to the equivalent solution. Note that both will result to valid combined second order compensation, but no longer lag-lead compensators, as in the example for **C2**).

Therefore, when correct parameters are used for Eq. (21) or Eqs. (13-14), they will produce correct results. However, those equations are very sensitive to small changes in their input parameters (and they should be). Based on this observation, we suggest that the graphically determined parameters $(\Omega_1, \Omega_2, \theta, \omega_1)$ are not proper for Eqs. (21).

Actually, the phase and gain margins from the solution in [4] can be found to be,

$$GM = 12.1727,\ PM = 41.8735°$$

Indeed, they are very close to the original design specification and for all practical purposes, sufficiently accurate. It is interesting to know that when applying either new gain margin $GM = 12.1727$ to design procedure for **B1**) or new phase margin $PM = 41.8735°$ to design procedure for **B2**), a proper lag-lead compensator can be found which is very close to the solution obtained in [4]. For example, using $GM = 12.1727$ and frequencies $\omega_1 = 20.65$ and $\omega_2 = 9$ into design procedure for **B1**), we find,

$$[\tau, \sigma, \alpha, \beta] = [0.0273\ \ 15.9489\ \ 13.9865\ \ 0.0715]$$

which are very close to the solution in [4]. However, $GM = 12$ will produce a result similar to that from the design example for specification **C2**). Actually, a detailed numerical analysis indicates that there is no three-parameter lag-lead compensator in the form of Eq. (1) which can meet $GM = 12\ dB$ and $PM = 42°$ at frequency $9\ rad$.

In this sense, the non-trial-and-error approach might lead back to a trial-and-error use in actual implementation when no valid lag-lead compensators can be found due to poor accuracy in determining $(\Omega_1, \Omega_2, \theta, \omega)$ from the universal design chart for lag-lead compensators.

VI. APPLICATIONS IN OPTIMAL DESIGNS

Since all steps can be carried out analytically in the design procedures we have discussed in the previous sections, it is easy to utilize those procedures for the purpose of various optimal lag-lead compensation. In this section, we present two simple optimal design problems:

M1) *Design a lag-lead compensator for the specified gain margin GM at phase crossover frequency* ω_1, *and maximize phase margin within the specified range of gain crossover frequency* $[\omega_{2\min}, \omega_{2\max}]$;

M2) *Design a lag-lead compensator for the specified phase margin PM at phase crossover frequency* ω_2, *and maximize the gain margin within the specified range of phase crossover frequency* $[\omega_{1\min}, \omega_{1\max}]$.

Specifications **M1**) and **M2**) are based directly on **C1**) and **C2**), the corresponding design procedures can be constructed similarly. It should be pointed out here that the maximum phase or gain does not necessary imply a better performance with respect to many measures in time domain.

The design procedure for specification **M1**) can be outlined as following,

Procedure M1:

M1-1) Calculate c_1, and δ_1;

$$c_1 = \frac{10^{-GM/20}}{K_c\left|G_p(j\omega_1)\right|},\ p_1 = -\angle G_p(j\omega_1) - 180°;$$

M1-2) Search for ω_2; for any ω_2 in $[\omega_{2\min}, \omega_{2\max}]$, calculated c_2 using Eq. (9) and δ_2 using Eq. (10), and then find the corresponding phase margin as,

$$PM(\omega_2) = 180° + \angle G_p(j\omega_2) + p_2(\omega_2)$$

Search ω_2 that minimizes the error in the phase margin

$$\omega_2 = \arg\left\{\min_{\omega_{2\min} < \omega_2 < \omega_{2\max}} PM(\omega_2)\right\}$$

The search process can be carried out using *Fmin* again.

M1-3) to **M1-5**) is the same as **C1-3**) to **C1-5**).

The design procedure for specification **M2**) is,

Procedure M2:

M2-1) Calculate c_2, and δ_2;

$$c_2 = \frac{1}{K_c\left|G_p(j\omega_2)\right|},\ p_2 = -\angle G_p(j\omega_2) - 180°;$$

M2-2) Search for ω_1; for any ω_1 in $[\omega_{1\min}, \omega_{1\max}]$, calculated δ_1 using Eq. (15) and c_1 using Eq. (16), and then find the corresponding gain margin as,

$$GM(\omega_1) = -\frac{10^{-c_1/20}}{K_c\left|G_p(j\omega_1)\right|}$$

Search ω_1 that maximize the gain margin

$$\omega_1 = \arg\left\{\min_{\omega_{1\min} < \omega_1 < \omega_{1\max}} \left|GM(\omega_1)\right|\right\}$$

M2-3) to **M2-5**) is same as M**1-3**) to M**1-5**).

Generally, specification **M2**) has no much practical use. To illustrate the design procedure for specification **M1**), we consider the same example as in the previous section.

***Design Example for Specification* M1)**

In this case, the design problem is given as: find a lag-lead compensator for the specified gain margin $GM = 12.5\ dB$ at phase crossover frequency $\omega_1 = 20.00\ rad/s$ and maximize phase margin within the specified range of gain crossover frequency $[5\ rad/s\ \ 10\ rad/s]$.

Step 1): It is easy to find that from Eq. (9),

$$c_1 = 0.4373,\ \delta_1 = 1.1667;$$

Step 2): The maximum phase margin and the corresponding gain crossover frequency are found to be,

$$PM = 34.1554°,\ \omega_2 = 8.6500,$$

and the corresponding $c_2 = 0.2285$, $\delta_2 = 0.9995$.

Step 3): Δ_1, Δ_2, Γ is calculated from Eq. (6),

$$\Delta_1 = -2.1550,\ \Delta_2 = -5.1891,\ \Gamma = 0.1305;$$

Step 4) and *Step* 5): the desired compensator is found to be,

$$(\tau, \sigma, \alpha, \beta) = (11.7429\ \ 0.0235\ \ 0.1131\ \ 8.8416);$$

and the corresponding transfer function is,

$$\bar{G}_c(s) = \frac{(s+0.729)(s+4.8208)}{(s+0.1334)(s+50.0866)} = \frac{s^2+5.5737s+3.6297}{s^2+42.7084s+3.6297}$$

Fig. 8 shows phase margin as a function of gain crossover frequency in this case, and Fig. 9 presents the Bode plot for the compensated system.

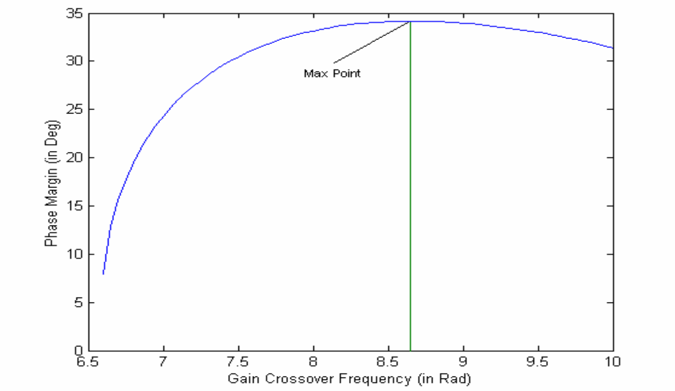

Fig. 8: Phase Margin as a Function of Frequency for Specification **M1**)

VII. CONCLUSIONS

A new and analytical non-trial-and-error approach for designing a special class of three-parameter phase lag-lead compensators has been developed in this paper. Since the new procedure can be carried out completely using computer programs without manual graphical operations, it can be used to simplify and enhance the current teaching and industrial practices for lag-lead compensator design. For example, more sophisticated design and performance analysis, such as optimal design and sensitivity analysis, can now be conducted easily with the new design method, which is not practical with traditional graphic-based trial-and-error design method or even graphic-based non-trial-and-error approach, since the number of trial-and-errors or manual manipulations required in those cases is simply too large. The new design method provides a simply, straightforward, and systematic approach for teaching this subject to students. Discussion on more advanced design method for similar control problems with stability and robustness consideration can be found in textbooks [22-23].

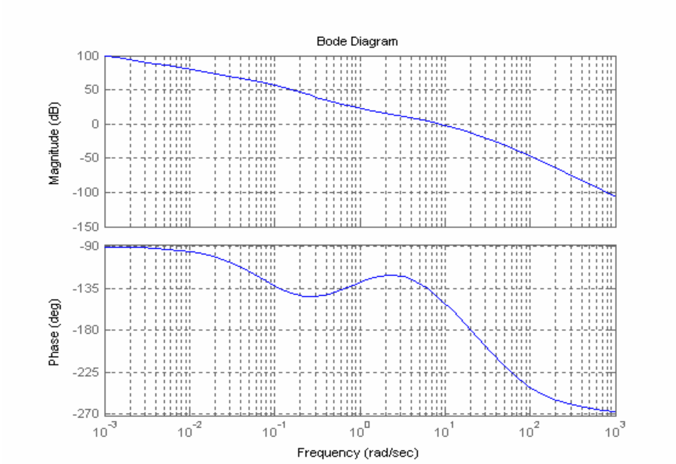

Fig. 9: Bode Plots for Compensated Plant in Specification **M1**)

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APPENDIX: SOLUTION TO EQUATION (5)

As in [8], let $\mu = \Delta(\omega)\Gamma$, then, from Eq. (5)

$$\frac{1+\mu^2}{1+\Delta^2} = c^2,\ \Delta = \frac{\mu-\delta}{1+\mu\delta}$$

and

$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

is used in the process.

Eliminating Δ from the first equations, we have

$$(1+\mu^2)\left[(1+\mu\delta)^2 - c^2(1+\delta^2)\right] = 0$$

Since μ must be a real number, thus,

$$\mu = \frac{\pm c\sqrt{1+\delta^2}-1}{\delta},\ \Delta = \frac{c \mp \sqrt{1+\delta^2}}{c\delta}$$

hence

$$\Gamma = \frac{\mu}{\Delta} = \frac{c\left(\pm c\sqrt{1+\delta^2}-1\right)}{c \mp \sqrt{1+\delta^2}}$$

Since τ, σ, α, β are all positive, Γ must be positive too, therefore, only the positive sign in μ is valid, thus the final solution is,

$$\Gamma = \frac{c\left(c\sqrt{1+\delta^2}-1\right)}{c-\sqrt{1+\delta^2}},\ \Delta = \frac{c-\sqrt{1+\delta^2}}{c\delta}.$$

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Fei-Yue Wang received Ph.D. in Electrical, Computer and Systems Engineering from the Rensselaer Polytechnic Institute, Troy, New York in 1990. He jointed the University of Arizona in 1990 and currently is a Professor and the Director of the Program for Advanced Research in Complex Systems. In 1999, he found the Intelligent Control and Systems Engineering Center at the Institute of Automation, Chinese Academy of Sciences, Beijing, China, under the support of the Outstanding Oversea Chinese Talents Program. Since 2002, he has been the Director of the Key Laboratory of Complex Systems and Intelligence Science at the Chinese Academy of Sciences. His current research interests include modeling, analysis, and control mechanism of complex systems; agent-based control systems; intelligent control systems; real-time embedded systems, application specific operating systems (ASOS); applications in intelligent transportation systems, intelligent vehicles and telematics, web caching and service caching, smart appliances and home systems, and network-based automation systems. He has published more than 200 book, book chapters, and papers in those areas since 1984 and received more than $20M USD and over ¥50M RMB from NSF, DOE, DOT, NNSF, CAS, MOST, Caterpillar, IBM, HP, AT&T, GM, BHP, RVSI, ABB, and Kelon. He received Caterpillar Research Invention Award with Dr. P.J.A. Lever in 1994 for his work in robotic excavation and the National Outstanding Young Scientist Research Award from the National Natural Science Foundation of China in 2001, as well as various industrial awards for his applied research from major corporations. He was the Editor-in-Chief of the *International Journal of Intelligent Control and Systems* from 1995 to 2000, Editor-in-Charge of *Series in Intelligent Control and Intelligent Automation* from 1994 to 2004, and currently is the Editor-in-Charge of *Series in Complex Systems and Intelligence Science*, ITS Department Editor and Associate Editor of the *IEEE Intelligent Systems*, and Associate Editor of the *IEEE Transactions on Systems, Man, and Cybernetics*, *IEEE Transactions on Robotics and Automation, IEEE Transactions on Intelligent Transportation Systems*, *IEEE Transactions on Knowledge and Data Engineering*, and several other international journals. He was an elected member of IEEE SMC Board of Governors and IEEE ITSC AdCom, IEEE Nano Tech Council, and is the President of IEEE Intelligent Transportation Systems Society. He was the Program Chair of the 1998 IEEE Int'l Symposium on Intelligent Control, the 2001 IEEE Int'l Conference on Systems, Man, and Cybernetics, Chair for Workshops and Tutorials for 2002 IEEE Int'l Conf. on Decision and Control (CDC), the General Chair of the 2003 IEEE Int'l Conference on Intelligent Transportation Systems, Co-Program Chair of the 2004 IEEE Int'l Symposium on Intelligent Vehicles and the General Chair for the same conference in 2005, the General Chair of 2005 ASME/IEEE Int'l Conference on Mechatronic/Embedded Systems and Applications, and the Program Chair of IEEE Int'l Conf. on Vehicular Electronics and Safety . He was the President of Chinese Association for Science and Technology , USA, and the Vice President and one of the major contributors of the American Zhu Kezhen Education Foundation. Dr. Wang is also a member of Sigma Xi, ACM, AMSE, ASEE, and a Fellow of the International Council of Systems Engineering (INCOSE).